MATH 208 C — MIDTERM 2 — Autumn 2022

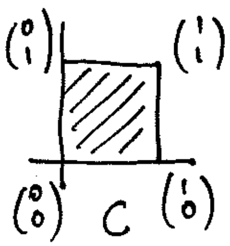
NAME: Solutions

To make it possible for Gradescope to recognize you, please write your name:

(1) **clearly** in **CAPITAL LETTERS**,
(2) **exactly on the line above**,
(3) use the **name you are registered under** for this class.

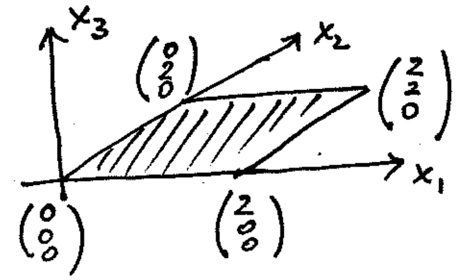
(1) Please put away all phones and earphones in your bag.
(2) There are 4 problems.
(3) Show all of your work and justify your answers.
(4) Write clearly.

(1) (a) Let C be the unit square in $\mathbb{R}^2$ with corners $(0,0),(1,0),(0,1),(1,1)$. What is the image of C under the linear transformation $T(x) = Ax$ where
$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}?$$
Draw a picture of the image and mark everything clearly.



$$T\left(\begin{pmatrix}1\\0\end{pmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\0\\0\end{pmatrix} \qquad T\left(\begin{pmatrix}0\\1\end{pmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\2\\0\end{pmatrix}$$

$\therefore$ C goes to the square in $\mathbb{R}^3$ w/



corners $\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}2\\0\\0\end{pmatrix}, \begin{pmatrix}0\\2\\0\end{pmatrix}, \begin{pmatrix}2\\2\\0\end{pmatrix}$

since $T\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$ for any linear transformation

(b) Write down the linear transformation that will take C and **first** apply T to it and **then reflect** the image of C under T across the $(\mathbf{x}_2, \mathbf{x}_3)$-plane.
The transformation should be written in full with domain, codomain and matrix. The matrices in the composition must be written out but you don't need to multiply them out.

Let $R: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be reflection across (x_2, x_3)-plane.

Then $R\left(\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}\right) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ since $\begin{pmatrix}1\\0\\0\end{pmatrix} \longmapsto \begin{pmatrix}-1\\0\\0\end{pmatrix}$ but $e_2 \longmapsto e_2$ $e_3 \longmapsto e_3$

The composed transformation is $R \circ T$

$R \circ T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$$\begin{pmatrix}x_1\\x_2\end{pmatrix} \longmapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{pmatrix}x_1\\x_2\end{pmatrix}$$

(2) (a) Is the following a linear transformation? Give reasons for your answer.

$$T : \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ \begin{pmatrix} x \\ y \end{pmatrix} & \mapsto & \begin{pmatrix} 2x+y \\ x+1 \end{pmatrix} \end{array}$$

Ans 1 $\quad T\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \therefore$ T is not a linear transf.

Ans 2 $\quad T\left(c\begin{pmatrix} x \\ y \end{pmatrix}\right) = T\begin{pmatrix} cx \\ cy \end{pmatrix} = \begin{pmatrix} c(2x+y) \\ cx+1 \end{pmatrix} \neq c\, T\begin{pmatrix} x \\ y \end{pmatrix}$

Ans 3 : $\quad T\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) = \cancel{T(x_1} \; T\left(\begin{pmatrix} x_1+x_2 \\ y_1+y_2 \end{pmatrix}\right) = \begin{pmatrix} 2x_1+y_1+2x_2+y_2 \\ x_1+x_2+1 \end{pmatrix}$

$\neq T\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + T\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$

(b) Is the following a vector space? Give reasons for your answer.

$$S = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : y = x^2 \right\}.$$

No : A point in S looks like $\begin{pmatrix} x \\ x^2 \end{pmatrix}$

for $c \in \mathbb{R}$ $\quad c\begin{pmatrix} x \\ x^2 \end{pmatrix} = \begin{pmatrix} cx \\ cx^2 \end{pmatrix}$ but $cx^2 \neq (cx)^2$

So this set fails to be closed under scalar mult$^{\underline{n}}$.

(3) Let T be the linear transformation such that $T(x) = Ax$ where A and its echelon form B are shown below.

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 0 & 4 \\ -1 & 6 & -8 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$ pivots

(a) Compute a basis for the kernel of T.

kernel $(T) = \text{nullsp}(A) = \text{nullsp}(B)$

Solve $Bx = 0$ $\quad x_2 = x_3 \quad x_1 = -3x_2 + x_3 = -2x_3$

$\therefore \text{Null}(B) = \left\{ \begin{pmatrix} -2s \\ s \\ s \end{pmatrix} : s \in \mathbb{R} \right\} \Rightarrow$ basis of kernel (T)

$= \left\{ \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right\}$

(b) Compute a basis for the range of T.

range $(T) = \text{colsp}(A)$. The LI cols of A correspond to the pivot cols of B

$\therefore$ basis of range $(T) = \left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 6 \end{pmatrix} \right\}$

(c) Is T invertible? If yes, compute T^{-1}. If not, say why not.

No since T is invertible $\Leftrightarrow$ A is invertible

$\Leftrightarrow$ A can be row reduced to I

But B shows that the reduced echelon form of A cannot be I.

(4) In the following questions, all transformations must be written fully with domain, codomain and matrix. Let

$$S = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} : \begin{array}{r} x + 2y + 3z - w = 0 \\ -z + w = 0 \end{array} \right\}.$$

(a) What is the dimension of S?

dimension of S is 2 since it is the intersection of 2 3d planes in $\mathbb{R}^4$. Alternately there are 2 free vars in the system

(b) Find a linear transformation P such that $S = kernel(P)$.

$$P : \mathbb{R} \longrightarrow \mathbb{R}$$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \longmapsto \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

$$kernel(P) = nullsp\left(\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \right) = S$$

(c) Find a linear transformation Q such that $S = range(Q)$.

Solve the eqns:

$$z = w$$
$$x = -2y - 3z + w = -2y - 3w + w$$
$$= -2y - 2w$$

$$S = \left\{ \begin{pmatrix} -2s - 2t \\ s \\ t \\ t \end{pmatrix} : s, t \in \mathbb{R} \right\} = Span\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$$Q : \mathbb{R}^2 \longrightarrow \mathbb{R}^4$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{bmatrix} -2 & -2 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

MATH 208 D — MIDTERM 2 — Autumn 2022

NAME: Solutions

To make it possible for Gradescope to recognize you, please write your name:

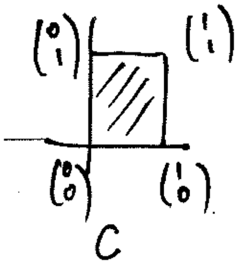
(1) **clearly** in **CAPITAL LETTERS**,
(2) **exactly on the line above**,
(3) use the **name you are registered under** for this class.

(1) Please put away all phones and earphones in your bag.
(2) There are 4 problems.
(3) Show all of your work and justify your answers.
(4) Write clearly.

(1) (a) Let C be the unit square in $\mathbb{R}^2$ with corners: $(0,0),(1,0),(0,1),(1,1)$. What is the image of C under the linear transformation $T(x) = Ax$ where

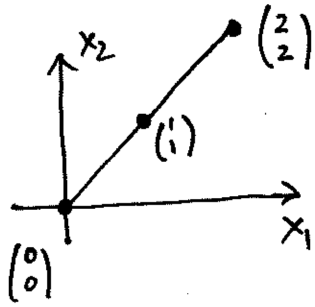
$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}?$$

Draw the image and mark it clearly.



$T\left(\begin{pmatrix}0\\0\end{pmatrix}\right) = \begin{pmatrix}0\\0\end{pmatrix}$ $\quad T\left(\begin{pmatrix}1\\0\end{pmatrix}\right) = \begin{pmatrix}1\\1\end{pmatrix}$ $\quad T\left(\begin{pmatrix}0\\1\end{pmatrix}\right) = \begin{pmatrix}1\\1\end{pmatrix}$

$\therefore\ T(C) =$



$T\left(\begin{pmatrix}1\\1\end{pmatrix}\right) = \begin{pmatrix}2\\2\end{pmatrix}$

line segment in $\mathbb{R}^2$ with end pts $\begin{pmatrix}0\\0\end{pmatrix}$ & $\begin{pmatrix}2\\2\end{pmatrix}$

(b) Write down the linear transformation that will **first** apply T to C and **then** **rotate** the image of C under T by 90 degrees in the **clockwise** direction. *The transformation must be written in full with domain, codomain and matrix. The matrices in the composition must be written out but you don't need to multiply them out.*

Let $R : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the rotation matrix

then $R\left(\begin{pmatrix}x_1\\x_2\end{pmatrix}\right) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$

The compound transformation is $R \circ T$

$R \circ T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$\begin{pmatrix}x_1\\x_2\end{pmatrix} \longmapsto \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$

(2) (a) Is there any value of a for which the following is a linear transformation? If yes, find them all. If not, say why not.

$$T: \quad \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x+y \\ a+1 \end{pmatrix}$$

T must send $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$T\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ a+1 \end{pmatrix} \quad \therefore \quad a = -1$ is needed.

Then $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x+y \\ 0 \end{pmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ is a lin. trans.

(b) Is the following a vector space? Give reasons for your answer.

$$S = \{x \in \mathbb{R}^3 : Ax = x\}.$$

Yes Check the 3 properties.

(1) $A\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \therefore \quad \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in S$

(2) $Ax_1 = x_1 \quad Ax_2 = x_2 \quad \therefore \quad A(x_1 + x_2) = x_1 + x_2$

for $x_1, x_2 \in S$

(3) If $x \in S$, $c \in \mathbb{R}$ $\quad Ax = x \Rightarrow Acx = cAx = cx$.

Alternately $Ax = x \Leftrightarrow Ax = Ix \Leftrightarrow (A-I)x = 0$

$S = \{x \in \mathbb{R}^3 : (A-I)x = 0\} = \text{Null}(A-I)$ which is a subspace.

(3) Consider the matrix:

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -2 \\ -1 & 6 & 7 \end{bmatrix}.$$

(a) Does A have an inverse? If yes, compute it. If no, explain why.

$$\underset{A}{\begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -2 \\ -1 & 6 & 7 \end{bmatrix}} \xrightarrow[R_3 \leftarrow R_1+R_3]{R_2 \leftarrow R_2-2R_1} \begin{bmatrix} 1 & 3 & 2 \\ 0 & -6 & -6 \\ 0 & 9 & 9 \end{bmatrix} \xrightarrow[R_3 \leftarrow \frac{1}{9}R_3]{R_2 \leftarrow -\frac{1}{6}R_2} \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

A cannot be reduced to I so A is <u>not</u> Invertible.

(b) Compute a basis for the range of the transformation $T(x) = Ax$. Explain.

range (T) = colsp (A). Looking at B the 1st & 2nd cols of A are LI

basis of range(T) = $\left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 6 \end{pmatrix} \right\}$

(c) What is meant by the nullity of A and what is it in this case? Explain.

nullity (A) = dim nullsp (A)

In this case A has 3 cols . rank (A) = 2 by (b)

$\therefore$ Nullity (A) = 3 − 2 = 1.

(4) Consider the set $S = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : ax_1 + bx_2 = 0 \right\}$ where $a \neq 0$ and $b \neq 0$.

(a) Find a linear transformation T such that $S = kernel(T)$. Write it completely, with domain, codomain, matrix etc.

$$kernel(T) = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : ax_1 + bx_2 = 0 \right\}$$

$$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^1$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \begin{bmatrix} a & b \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

(b) Express S as a span. Solve $ax_1 + bx_2 = 0$

$$S = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -b/a \, x_2 \\ x_2 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} -b/a \, s \\ s \end{pmatrix} : s \in \mathbb{R} \right\}$$

$$= Span \left\{ \begin{pmatrix} -b/a \\ 1 \end{pmatrix} \right\}$$

(c) Find a linear transformation U such that $S = range(U)$. Write it completely, with domain, codomain, matrix etc.

$$range(U) \text{ must be } Span \left\{ \begin{pmatrix} -b/a \\ 1 \end{pmatrix} \right\} = colsp \begin{pmatrix} \text{matrix} \\ \text{of } U \end{pmatrix}$$

$$U: \mathbb{R}^1 \longrightarrow \mathbb{R}^2$$

$$x \longmapsto \begin{bmatrix} -b/a \\ 1 \end{bmatrix} x$$